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Hamiltonian Formalism in Supermechanics

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The Hamilton–Cartan formalism in supermechanics is developed, the graded structure on the manifold of solutions of a variational problem defined by a regular homogeneous Berezinian Lagrangian density is determined and its graded symplectic structure is analyzed. The graded symplectic structure on the manifold of solutions of a classical regular Lagrangian is compared with the Koszul–Schouten brackets.

KEY WORDS: Batalin–Vilkoviski structures; Berezinian and Lagrangian density; graded symplectic structure; Koszul–Schouten bracket; Poincaré–Cartan form.

1. INTRODUCTION

The goal of this paper is to develop the Hamilton–Cartan formalism in supermechanics; i.e., for variational problems on the space of curves of a graded manifold. In this development, the first key point is to realize the important role that $\mathbb{R}^{1|1}$ plays in the graded setting: On the one hand the sections of the structure sheaf of a graded manifold $(M, \mathcal{A})$ can be recovered as the graded morphisms from $(M, \mathcal{A})$ into $\mathbb{R}^{1|1}$, and on the other, the natural integration theory of graded vector fields uses $\mathbb{R}^{1|1}$-flows, not $\mathbb{R}$-flows (see Monterde and Sánchez-Valenzuela, 1993). Consequently, we formulate variational problems on the space of $\mathbb{R}^{1|1}$-curves with values in a graded manifold. Up to our knowledge, this possibility has never been considered before. A $\mathbb{R}^{1|1}$-curve in $(M, \mathcal{A})$ is a graded morphism $\gamma\colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$. Such a morphism determines a footprint on the base manifolds $\bar{\gamma}\colon \mathbb{R} \to M$ which is nothing but a classical curve on M. Nevertheless, the $\mathbb{R}^{1|1}$-curve is more than just its footprint. It also has an important "soul" part.

The second key point is the use of the Berezin integral on the manifold $\mathbb{R}^{1|1}$ in order to state the variational problems. It is only with such a kind of integral

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that the Euler–Lagrange equations associated to a variational problem are the equations one can expect and where even and odd variables are treated on the same footing.

Once we have made these two choices ($\mathbb{R}^{1|1}$-curves and Berezin integral) the whole program of variational calculus works in a natural way. Several parts of this program have been developed already: The construction of graded jet bundles (Hernández Ruipérez and Muñoz Masqué, 1984, 1987), the deduction of the Euler–Lagrange equations for a Berezinian Lagrangian density (Monterde, 1992a), and the definition of regular Lagrangians as well as the very first steps of supermechanics (Monterde and Muñoz Masqué, 1992).

Here, we propose a new definition of Poincaré–Cartan form and we show that—as in the classical case—there is a bijection between the critical sections of a variational problem and the extremals of the Poincaré–Cartan form. In this setting, we introduce canonical coordinates, we compute the radical of the exterior derivative of the Poincaré–Cartan form, we write down the Hamilton equations in canonical coordinates, and we solve them. The final stage of this study is to define the graded manifold of solutions and a symplectic structure on it.

At this point some remarks should be done: The first one is that if the graded dimension of $(M, \mathcal{A})$ is (m, n), then the dimension of the graded manifold of solutions is $(2(m+n), 2(m+n))$. This shows that, even though in mechanics the manifold of solutions of a variational problem is the tangent bundle, this is no longer true in supermechanics: The graded manifold of solutions is not the supertangent bundle of $(M, \mathcal{A})$. In fact, the graded dimensions of the two notions of supertangent bundle appearing in the literature are $(2m, 2n)$ and $(2m+n, m+2n)$. There is an earlier approach to supermechanics proposed in Cariñena and Figueroa (1997). In this paper the authors use the right notion of supertangent, but, as a consequence of the previous remark, their approach cannot be the same than the one proposed here.

The second remarkable fact is the change of parity. If the initial graded Lagrangian function is even (resp. odd), then the resulting symplectic form is odd (resp. even). This means that if one wants to use a classical Lagrangian to define a graded variational problem, then the simplectic graded form will be an odd symplectic form. Therefore, given a regular classical Lagrangian function L on a differentiable manifold M, we can define a variational problem on the graded manifold $(M, C^\infty(M))$. By applying all our previous results, we construct a graded manifold of solutions $(S, \mathcal{A}_S)$ together with a variational symplectic form ω_S.

Moreover, according to the classical variational calculus, L defines a symplectic form on TM. Such a symplectic form induces an odd symplectic structure on the graded manifold (TM, Ω_{TM}). This odd symplectic structure is called the Koszul–Schouten symplectic structure and its graded Poisson bracket is a particular case of Batalin–Vilkovisky structure. Our last result is to show that the graded symplectic manifolds (TM, Ω_{TM}) together with the Koszul–Schouten symplectic form ω_K and the manifold of solutions $(S, \mathcal{A}_S)$ together with the variational symplectic form ω_S, are isomorphic. We thus conclude that the Koszul–Schouten symplectic structures can be obtained by simply adapting variational calculus to the graded manifold category.

2. CURVES ON GRADED MANIFOLDS

We work in the category of C^∞ graded manifolds; definitions are taken from Kostant (1977). On a graded manifold $(M, \mathcal{A})$ of graded dimension (m, n) positive indices are used for even coordinates: x^i, $i = 1, \ldots, m$, and negative indices for odd coordinates: x^i, $i = -n, \ldots, -1$. The natural homomorphism is denoted by $\mathcal{A} \to C^\infty_M$, $f \mapsto \tilde{f}$. Nevertheless, the coordinates of the graded manifold $\mathbb{R}^{1|1}$—with base manifold $\mathbb{R}$ and graded ring $\mathcal{R}^{1|1}$—are denoted by (t, s), with $|t| = 0$, $|s| = 1$; i.e., $\mathcal{R}^{1|1} = \{f(t) + g(t)s : f, g \in C^\infty(\mathbb{R})\}$. In the category of graded manifolds, $\mathbb{R}^{1|1}$ plays the same role as $\mathbb{R}$ in the category of differentiable manifolds. This is due to the following two basic facts: (1) The graded morphisms between $(M, \mathcal{A})$ and $\mathbb{R}^{1|1}$ are exactly the global sections of the structure sheaf; i.e., $\mathrm{Mor}((M, \mathcal{A}), \mathbb{R}^{1|1}) = \mathcal{A}(M)$, and (2) a theory of integration for graded vector fields is only possible if one uses $\mathbb{R}^{1|1}$-flows, but not with $\mathbb{R}$-flows (cf. Monterde and Sánchez-Valenzuela, 1993).

We recall that a classical curve $\gamma\colon \mathbb{R} \to M$ can be seen as a section of $p_1\colon \mathbb{R} \times M \to \mathbb{R}$. In the graded case we must substitute $\mathbb{R}^{1|1}$ for $\mathbb{R}$. Hence a graded curve must be a section of the graded submersion $p_1\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ given by the projection onto the first factor, or equivalently, a morphism of graded manifolds $\gamma\colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$.

We denote by $d\colon \Omega^r(M) \to \Omega^{r+1}(M)$ the exterior differential on a classical differentiable manifold M and by $\mathrm{d}\colon \Omega^r(M, \mathcal{A}) \to \Omega^{r+1}(M, \mathcal{A})$ the graded exterior differential on a graded manifold $(M, \mathcal{A})$.

Example 2.1. In order to work out an example, let us choose a particular graded manifold. Let M be a differentiable manifold and let us consider the graded manifold (M, Ω_M), where Ω_M denotes the sheaf of differential forms on M. Hence $\dim(M, \Omega_M) = (m, m)$ if $m = \dim M$. Given a coordinate system $(y^1, \ldots, y^m)$ on M, we can build up a system of adapted graded coordinates: (y^i, dy^i). According to our way of denoting graded coordinates, (x^i), $i = -m, \ldots, -1, 1, \ldots, m$, we have $x^i = y^i$, $x^{-i} = \mathrm{d}y^i$, $i = 1, \ldots, m$. A graded curve $\gamma\colon \mathbb{R}^{1|1} \to (M, \Omega_M)$ is determined by a pair of maps $\gamma\colon \mathbb{R} \to M$, $\gamma^*\colon \Omega(M) \to \mathcal{R}^{1|1}$. Note that the homomorphism γ^* is not necessarily the pull-back map of $\gamma\colon \mathbb{R} \to M$. We have

$$\left.\begin{aligned} \gamma^*(y^i) &= y^i \circ \gamma = f^i(t), \\ \gamma^*(dy^i) &= g^i(t)s, \end{aligned}\right. \qquad f^i, g^i \in C^\infty(\mathbb{R});\ i = 1, \ldots, m. \tag{1}$$

If γ^* is the pull-back of $\gamma\colon \mathbb{R} \to M$, then $g^i = (f^i)'$. Also note the following—in principle—disappointing fact: If $\alpha \in \Omega(M)$ is a differentiable form without 0 and 1-degree parts, then $\gamma^*(\alpha) = 0$.

3. GRADED FIRST-ORDER JET BUNDLE

The usual coordinate description of jet bundles does not work with graded manifolds. A more intrinsic construction of graded jet bundles, entailing further algebraic formalizations, is needed. This is done in Hernández Ruipérez and Muñoz Masqué (1984) and we do not repeat it here. According to this construction we can define the graded 1-jet bundle $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ of local sections of $p_1 \colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$; its graded dimension is $(1 + 2m + n, 1 + m + 2n)$, $\dim(M, \mathcal{A}) = (m, n)$. We remark that the base manifold underlying $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is not equal to $J^1(\mathbb{R}, M)$. The graded ring of $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is denoted by $\mathcal{A}^1$. Also we denote the graded ring of $\mathbb{R}^{1|1} \times (M, \mathcal{A})$ by $\mathcal{A}^0$ as $J^0(\mathbb{R}^{1|1}, (M, \mathcal{A})) = \mathbb{R}^{1|1} \times (M, \mathcal{A})$. The graded fibred coordinates on first-order jet bundles are defined in Hernández Ruipérez and Muñoz Masqué (1984) and Monterde (1992a); we denote them by $(t, s, x^i; x^i_t; x^i_s)$, $i = -n, \ldots, -1, 1, \ldots, m$, with $|t| = |x^i| = |x^i_t| = |x^h_s| = 0$, and $|s| = |x^h| = |x^h_t| = |x^i_s| = 1$, for $h = -n, \ldots, -1; i = 1, \ldots, m$.

We remark that the intrinsic algebraic construction of graded jet bundles can produce shocking facts like: $j^1(\gamma)^*(x^i_s) = 0$ for every local section $\gamma \colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|1} \times (M, \mathcal{A})$, when $i > 0$.

3.1. Curves and the First-Order Jet Bundle

As is well known, a variation of a classical curve $\gamma \colon \mathbb{R} \to M$ is a 1-parameter family of curves $\overline{\gamma_{\overline{t}}}(t)$ ($\overline{t} \in \mathbb{R}$ being the variational parameter) such that $\overline{\gamma_0} = \gamma$. According to our philosophy of substituting $\mathbb{R}^{1|1}$ for $\mathbb{R}$, a variation of a graded curve on a graded manifold, $\gamma \colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$, must also be a $\mathbb{R}^{1|1}$-parameter family of graded curves.

We exclusively consider variations of a curve induced by a graded vector field. In the classical case, the variations of a curve induced by a vector field are just the composition of the curve with the integral flow of the vector field. It can be shown that any even graded vector field can be integrated by simply using an even parameter, but the situation is different in the odd case.

Let us briefly recall the problem of existence and uniqueness of solutions of first-order superdifferential equations that have been studied in Monterde and Sánchez-Valenzuela (1993). The first fact to note is that the parameter space in the graded setting is $\mathbb{R}^{1|1}$ and that the problem of founding the integral flow of a graded vector field must be stated in terms of this parameter space. Second, once we have chosen the parameter space, we must choose a model of graded vector field on it. It is easy to check that there are three possible graded Lie algebra structures on $\mathbb{R}^{1|1}$ each giving rise to a different model of right-invariant graded vector field. For example, for the additive structure (the one we use in what follows) the model of graded vector field is given by $\partial/\partial t + \partial/\partial s$.

Let X be a graded vector field on the graded manifold $(M, \mathcal{A})$. We say that $\Gamma\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to (M, \mathcal{A})$ is the flow of X if together with an initial condition the following equation holds:

$$ev_{t=0} \circ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \circ \Gamma^* = ev_{t=0} \circ \Gamma^* \circ X,$$

where $ev_{t=0}$ is the map defined by $ev_{t=0}(f(t) + g(t)s) = f(0)$. In Monterde and Sánchez-Valenzuela (1993), it is shown that any graded vector field can be integraded—in the previous sense—by means of integral curves parametrized on $\mathbb{R}^{1|1}$. It is also shown there that if the homogeneous parts X_0, X_1 of X satisfy the equations $[X_0, X_1] = [X_1, X_1] = 0$, then the previous equation also holds without the evaluation map; i.e.,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \circ \Gamma^* = \Gamma^* \circ X. \tag{2}$$

Moreover the flow induces an action of the additive Lie group structure of $\mathbb{R}^{1|1}$ on the supermanifold $(M, \mathcal{A})$ and then, a kind of relation like $\Gamma_{t_1,s_1} \circ \Gamma_{t_2,s_2} = \Gamma_{t_1+t_2,s_1+s_2}$ is valid.

Example 3.1. Let us come back to the graded manifold (M, Ω_M). Graded vector fields on it are derivations of the graded algebra Ω_M. For example, given a vector field on M, the Lie derivative, $\mathcal{L}_X$, is an even graded vector field on (M, Ω_M). The integral flow of this graded vector field is given by $\Phi^*_{\bar t}$, the pull-back of the integral flow $\Phi_{\bar t}$ of X. No odd parameter is needed in order to integrate even vector fields. Therefore the variation of a graded curve produced by the graded vector field $\mathcal{L}_X$ is given by

$$\begin{aligned} \gamma^* \circ \Phi^*_{\bar t}(y^i) &= \gamma^*(y^i \circ \Phi_{\bar t}) = f^i(t, \bar t), \\ \gamma^* \circ \Phi^*_{\bar t}(dy^i) &= \gamma^*(\mathrm{d}y^i \circ \Phi_{\bar t}) = g^i(t, \bar t)s. \end{aligned} \qquad i = 1, \ldots, m.$$

Example 3.2 (cf. Monterde and Sánchez-Valenzuela, 1993). On the graded manifold (M, Ω_M) the exterior derivative d is an example of an odd vector field. Its integral flow is given by the map $\Gamma = (\Gamma, \Gamma^*) : \mathbb{R}^{1|1} \times (M, \Omega_M) \to (M, \Omega_M)$, with $\Gamma = \pi_M\colon \mathbb{R} \times M \to M$ and $\Gamma^*\colon \Omega(M) \to \mathcal{A}_{\mathbb{R}^{1|1} \times (M, \Omega_M)}$ is given by $\Gamma^*(\alpha) = \alpha + \bar s d\alpha$, where $\bar t$, $\bar s$ are the graded coordinates in $\mathbb{R}^{1|1}$. Therefore the variation of a graded curve (1), produced by the graded vector field d, is given by

$$\begin{aligned} \gamma^* \circ \Gamma^*(y^i) &= \gamma^*(y^i + \bar s dy^i) = f^i(t) + g^i(t)\bar s s, \\ \gamma^* \circ \Gamma^*(\mathrm{d}y^i) &= \gamma^*(dy^i) = g^i(t)s. \end{aligned} \qquad i = 1, \ldots, m. \tag{3}$$

Example 3.3. Let us describe the 1-jet prolongation of a graded curve γ on (M, Ω_M). If γ^* is given by (1), then $j^1\gamma$ is determined by the following equations:

$$\begin{aligned} j^1(\gamma)^*\left(x_t^i\right) &= \frac{\partial}{\partial t}(f^i(t)) = (f^i)'(t), \\ j^1(\gamma)^*\left(x_s^i\right) &= \frac{\partial}{\partial s}(\gamma^*(y^i)) = 0, \\ j^1(\gamma)^*\left(x_t^{-i}\right) &= \frac{\partial}{\partial t}(g^i(t)s) = (g^i)'(t)s, \\ j^1(\gamma)^*\left(x_s^{-i}\right) &= \frac{\partial}{\partial s}(g^i(t)s) = g^i(t). \end{aligned} \tag{4}$$

Moreover, for the curves (3), given by the variations produced by the graded vector field d, we have

$$\begin{aligned} j^1(\gamma_{\overline{t},\overline{s}})^*\left(x_t^i\right) &= \frac{\partial}{\partial t}(f^i(t) + g^i(t)\overline{s}s) = (f^i)'(t) + g^i(t))'\overline{s}s, \\ j^1(\gamma_{\overline{t},\overline{s}})^*\left(x_s^i\right) &= \frac{\partial}{\partial s}(\gamma^*_{\overline{t},\overline{s}}(y^i)) = \frac{\partial}{\partial s}(f^i(t) + g^i(t)\overline{s}s) = -g^i(t)\overline{s}, \\ j^1(\gamma_{\overline{t},\overline{s}})^*\left(x_t^{-i}\right) &= \frac{\partial}{\partial t}(g^i(t)s) = (g^i)'(t)s, \\ j^1(\gamma_{\overline{t},\overline{s}})^*\left(x_s^{-i}\right) &= \frac{\partial}{\partial s}(g^i(t)s) = g^i(t). \end{aligned} \tag{5}$$

Let us recall that for $i > 0$ we have $j^1(\gamma)^*(x_s^i) = 0$ for any curve γ. This fact could induce to think that the coordinate x_s^i is useless in the 1-jet bundle. Why to work with an algebraic construction of graded jet bundles which produces graded coordinates that vanish when evaluated at any curve? The reason is now clear after (5). From the second equation in (5) we see that this is no longer true for the $\mathbb{R}^{1|1}$-variations of curves. This fact shows that such a coordinate is needed.

4. VARIATIONAL PROBLEMS IN SUPERMECHANICS

4.1. Berezinian Densities on $\mathbb{R}^{1|1}$

Below we recall the construction of the Berezinian sheaf of $\mathbb{R}^{1|1}$ (for the general case, see Hernández Ruipérez and Muñoz Masqué, 1985; Monterde, 1992a). Let $P^1(\mathbb{R}^{1|1})$ (resp. $\Omega^1_{\mathbb{R}^{1|1}}$) be the sheaf of differential operators of order ≤ 1 (graded 1-forms) on $\mathbb{R}^{1|1}$. We have $\mathrm{Ber}(\mathbb{R}^{1|1}) = \Omega^1_{\mathbb{R}^{1|1}} \otimes P^1(\mathbb{R}^{1|1})/K_1$, where K_1 is the right $\mathcal{R}^{1|1}$-submodule of the operators P such that for every $f \in \mathcal{R}^{1|1}$ with compact support, there exists an ordinary 0-form with compact support $g \in \mathcal{R}^{1|1}$ satisfying $dg = P(f)\tilde{}$. We denote by $[P]$ the coset of $P \in \Omega^1_{\mathbb{R}^{1|1}} \otimes P^1(\mathbb{R}^{1|1})$ in $\mathrm{Ber}(\mathbb{R}^{1|1})$. A basis of $\mathrm{Ber}(\mathbb{R}^{1|1})$ is then given by $[\mathrm{d}t \otimes \partial/\partial s]$. The Berezinian integral is defined as follows:

$$\int_{\mathrm{Ber}} \xi = \int_{\mathbb{R}} P(1)\tilde{}.$$

where $\xi = [P] \in \mathrm{Ber}(\mathbb{R}^{1|1})$ is a section with compact support. In the graded setting there is another kind of integration: The graded integral. For every graded 1-form ω on $\mathbb{R}^{1|1}$ with compact support we set:

$$\int_{\mathbb{R}^{1|1}} \omega = \int_{\mathbb{R}} \tilde{\omega},$$

where $\tilde{\omega}$ is the image of ω in the canonical homomorphism $\Omega^1_{\mathbb{R}^{1|1}}(\mathbb{R}^{1|1}) \to \Omega^1(\mathbb{R})$. Consequently, for every graded function $f = f_0(t) + f_1(t)s \in C^\infty(\mathbb{R}) \otimes \Lambda^{\cdot}\mathbb{R}$ with compact support, we have

$$\int_{\mathbb{R}^{1|1}} \mathrm{d}t \cdot f = \int_{\mathbb{R}} f_0 \, \mathrm{d}t, \qquad \int_{\mathrm{Ber}} \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right] f = \int_{\mathbb{R}} f_1 \, \mathrm{d}t.$$

This shows that the graded integral $\int_{\mathbb{R}^{1|1}}$ integrates on the first component of f, while the Berezinian integral $\int_{\mathrm{Ber}}$ integrates on the last component of f.

4.2. The Sheaf of First-Order Berezinian Densities

Once we choose the integration procedure, we can state variational problems related with this integral, but first we need to define the variational objects that determine the variational problem. A construction, similar to that of the Berezinian sheaf, leads us to the sheaf $\mathrm{Ber}^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$: Its sections are of the form $[\mathrm{d}t \otimes \frac{d}{ds}]L$, with $L \in \mathcal{A}^1$, and $\frac{d}{ds}$ being the total (or horizontal) derivative with respect to s (see Monterde, 1992a). These sections are the objects that define a variational problem.

4.3. Action Functional

Every global section $\xi \in \mathrm{Ber}^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ gives rise to a functional $\mathbb{L}^\xi$ defined by the formula

$$\mathbb{L}^\xi(\gamma) = \int_{\mathrm{Ber}} (j^1\gamma)^*\xi = \int_{\mathrm{Ber}} (j^1\gamma)^* \left[\mathrm{d}t \otimes \frac{d}{ds}\right] L = \int_{\mathrm{Ber}} \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right] (j^1\gamma^* L)$$

on the space of the sections of $p_1\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ for which the integral above converges.

If L is even, then $\int_{\mathrm{Ber}}(j^1\gamma)^*\xi$ vanishes for any curve γ. Indeed,

$$\int_{\mathrm{Ber}}(j^1\gamma)^*\xi = \int_{\mathrm{Ber}}\left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right](j^1\gamma^* L) = 0,$$

since $j^1\gamma^* L \in \mathcal{R}^{1|1}$ is even and hence it is of the form $f(t)$; so $\partial/\partial s(f(t)) = 0$. As $\mathbb{R}$-variations of γ does not change the degree of $j^1\gamma^* L$, the action functional vanishes identically. This fact again evidences the need of using $\mathbb{R}^{1|1}$-variations of curves. When doing this, the expression under the Berezin integral becomes an element in $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$; i.e., the integrand depends on the four graded coordinates $\{t, s, \overline{t}, \overline{s}\}$.

Let $\Gamma \colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to (M, \mathcal{A})$ be the integral flow of a vector field X. The variation of a curve γ is nothing but the composition

$$\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \xrightarrow{(\mathrm{id}\times\gamma)} \mathbb{R}^{1|1} \times (M, \mathcal{A}) \xrightarrow{\Gamma} (M, \mathcal{A}).$$

For the sake of simplicity, let us denote this composition by $\gamma_{\overline{t},\overline{s}}$. Thus, $j^1\gamma^*_{\overline{t},\overline{s}}L$ is an element of the form $L_0(t, \overline{t})L_1(t, \overline{t})s + L_2(t, \overline{t})\overline{s} + L_3(t, \overline{t})s\overline{s}$ in $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$.

Remark 4.1. It can easily be shown that the variation by means of an odd vector field—e.g., the exterior derivative d on the graded manifold (M, Ω_M), Eq. (5)—of an action functional defined by a classical Lagrangian function, does not vanish; e.g., $L = \frac{1}{2}\sum_{i=1}^m (x_t^i)^2$.

4.4. The Variational Principle

In order to obtain critical sections, i.e., curves γ such that $\mathbb{L}^\xi(\gamma) = \int_{\mathrm{Ber}}(j^1\gamma)^*\xi$ takes the minimum for any variation produced by a vector field, we must compute the derivative of $\mathbb{L}^\xi(\gamma_{\overline{t},\overline{s}}) = \int_{\mathrm{Ber}}(j^1\gamma^*_{\overline{t},\overline{s}})\xi$ with respect to $\partial/\partial\overline{t} + \partial/\partial\overline{s}$, for this vector field is the one playing the role of $\partial/\partial\overline{t}$ in the classical integration problem of a vector field (see Section 3.1 and Monterde and Sánchez-Valenzuela, 1993). The derivative of the variation $\gamma_{\overline{t},\overline{s}}$ with respect to the model vector field $\partial/\partial\overline{t} + \partial/\partial\overline{s}$ on $\mathbb{R}^{1|1}$, evaluated at 0, is

$$\begin{aligned} ev|_{\overline{t}=0} \circ (\partial/\partial\overline{t} + \partial/\partial\overline{s}) \circ \gamma^*_{\overline{t},\overline{s}} &= ev|_{\overline{t}=0} \circ (\partial/\partial\overline{t} + \partial/\partial\overline{s}) \circ (\mathrm{id} \times \gamma^*) \circ \Gamma^* \\ &= ev|_{\overline{t}=0} \circ (\mathrm{id} \times \gamma^*) \circ (\partial/\partial\overline{t} + \partial/\partial\overline{s}) \circ \Gamma^* \\ &= \gamma^* \circ ev|_{\overline{t}=0} \circ (\partial/\partial\overline{t} + \partial/\partial\overline{s}) \circ \Gamma^* \\ &= \gamma^* \circ ev|_{\overline{t}=0} \circ \Gamma^* \circ X \\ &= \gamma^* \circ X, \end{aligned}$$

where we have used that $ev|_{\overline{t}=0} \circ \Gamma^* = \mathrm{id}$, by virtue of the initial condition of superdifferential equations.

In Monterde and Sánchez-Valenzuela (1993), it is shown that the usual relationship between Lie derivatives on forms, exterior differentiation, and interior multiplication, holds true in the theory of supermanifolds when one uses the right notion of integral flow of graded vectors fields. Thus, e.g., if L is an element of $\mathcal{A}$ and, $\gamma_{\bar{t},\bar{s}}$ denotes the integral flow of a graded vector field X, then

$$ev|_{\bar{t}=0} \circ (\partial/\partial\bar{t} + \partial/\partial\bar{s})\gamma^*_{\bar{t},\bar{s}}(L) = \gamma^*(X(L)) = \gamma^*(\mathcal{L}_X L).$$

The same holds when lifting to the first-order jet bundle. If $L \in \mathcal{A}^1$, then

$$ev|_{\bar{t}=0} \circ (\partial/\partial\bar{t} + \partial/\partial\bar{s}) j^1\gamma^*_{\bar{t},\bar{s}}(L) = j^1\gamma^*\big(X^{(1)}(L)\big) = j^1\gamma^*(\mathcal{L}_{X^{(1)}} L),$$

where $X^{(k)}$ denotes the prolongation of X to the k-th order jet bundle and $\mathcal{L}_{X^{(1)}}$ denotes the Lie derivative with respect to $X^{(1)}$ (see Monterde and Muñoz Masqué, 1992). Therefore, in our case, we obtain

$$\begin{aligned}
&ev|_{\bar{t}=0} \circ (\partial/\partial\bar{t} + \partial/\partial\bar{s}) \int_{\mathrm{Ber}} \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right] j^1\gamma^*_{\bar{t},\bar{s}}(L) \\
&= \int_{\mathrm{Ber}} \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right] ev|_{\bar{t}=0} \circ (\partial/\partial\bar{t} + \partial/\partial\bar{s}) j^1\gamma^*_{\bar{t},\bar{s}}(L) \\
&= \int_{\mathrm{Ber}} j^1\gamma^* \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right] \big(X^{(1)}(L)\big) \\
&= \int_{\mathrm{Ber}} j^1\gamma^* \left(\mathcal{L}_{X^{(1)}} \left[\mathrm{d}t \otimes \frac{\partial}{\partial s}\right](L)\right).
\end{aligned}$$

(For the definition of the Lie derivative of a Berezinian density see Hernández Ruipérez and Muñoz Masqué, 1987.) Therefore, given a section γ, we can define a linear functional $\delta_\gamma \mathbb{L}^\xi\colon \mathrm{Der}_c(\mathcal{A}) \to \mathbb{R}$, called the *first variation of* $\mathbb{L}^\xi$ *at* γ, where $\mathrm{Der}_c(\mathcal{A}) \subset \mathrm{Der}(\mathcal{A})$ is the ideal of vector fields with compact support, as follows:

$$\delta_\gamma \mathbb{L}^\xi(X) = \int_{\mathrm{Ber}} (j^1\gamma)^*\big(\mathcal{L}_{X^{(1)}}\xi\big).$$

A section γ is said to be a *Berezinian critical section* for the functional $\mathbb{L}^\xi$ if $\delta_\gamma \mathbb{L}^\xi = 0$.

5. EULER–LAGRANGE EQUATIONS

On the other hand, we can also define the concept of a critical section using the other way of integration: The graded integration. Every global section $\omega \in \Omega^1(J^2)$ of the sheaf of differential 1-forms on the second-order jet bundle gives rise to a functional $\mathbb{L}^\omega$. (The need of the shift to the second-order jet bundle will be clear after Theorem 5.1.) The functional is defined by the formula

$$\mathbb{L}^{\omega}(\gamma) = \int_{\mathbb{R}^{1|1}} (j^2\gamma)^*\omega$$

on the space of sections of $p_1\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ for which the above integral converges. Given a section γ, we can define a linear functional $\delta_\gamma \mathbb{L}^{\omega}\colon \operatorname{Der}_c(\mathcal{A}^0) \to \mathbb{R}$, called the *first variation of* $\mathbb{L}^{\omega}$ *at* γ, as follows:

$$\delta_\gamma \mathbb{L}^{\omega}(X) = \int_{\mathbb{R}^{1|1}} (j^2\gamma)^*(\mathcal{L}_{X^{(2)}}\omega).$$

A section γ is said to be a *graded critical section* for the functional $\mathbb{L}^{\omega}$ if $\delta_\gamma \mathbb{L}^{\omega} = 0$; i.e., if the first variation of $\mathbb{L}^{\omega}$ vanishes at γ. The fundamental fact is that a Berezinian variational problem is equivalent to a graded Lagrangian variational problem of higher order. The following theorem establishes such an equivalence:

Theorem 5.1 (*Comparison Theorem*, Monterde and Muñoz Masqué, 1992). *Let*

$$\xi = \left[\mathrm{d}t \otimes \frac{d}{ds}\right] L \in \operatorname{Ber}^1(\mathbb{R}^{1|1}, (M, \mathcal{A})), \quad L \in \mathcal{A}^1,$$

be a first-order Berezinian Lagrangian density, and let

$$\lambda_\xi = \mathrm{d}t \frac{dL}{ds} \in \Omega^1(J^2).$$

Then, for every section γ, we have $\delta_\gamma \mathbb{L}^{\xi} = \delta_\gamma \mathbb{L}^{\lambda_\xi}$. Consequently, we can associate in a canonical way an equivalent graded Lagrangian density λ_ξ to each Berezinian Lagrangian density ξ.

In this comparison result, the key point is that the link between Berezinian Lagrangian densities and graded ones, is given by the total derivative with respect to the odd coordinate. This will be recalled in the definition of Poincaré–Cartan forms.

5.1. Euler–Lagrange Equations of a Berezinian Density

Theorem 5.2 (Monterde, 1992a). *With the same notations as in the previous theorem, the Euler–Lagrange equations for the first-order Berezinian Lagrange density ξ are the following*:

$$(j^2\gamma)^*\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x^i_t}\right) - (-1)^{|x^i|}\frac{d}{ds}\left(\frac{\partial L}{\partial x^i_s}\right)\right) = 0,$$
$$i = -n, \ldots, -1, 1, \ldots, m.$$

Remark 5.3. It should be remarked that Berezinian variational problems are the only producing Euler–Lagrange equations with even and odd variables treated in the same equal footing. For graded variational problems, the corresponding Euler–Lagrange equations distinguish between even and odd coordinates.

Example 5.4. Let us consider a graded manifold with null odd dimension; i.e. (M, C^∞_M), and a classical first-order Lagrangian function on it, $L \in C^\infty(J^1(\mathbb{R}, M))$. Then, the equations above reduce to the classical Euler–Lagrange equations as the Lagrangian function does not depend on variables x^i_s.

Remark 5.5. Let us recall (cf. Section 4.3) that for an even Lagrangian, then $\mathbb{L}^\xi(\gamma)$ vanishes for all curves $\gamma\colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$. Nevertheless, when we consider $\mathbb{R}^{1|1}$-variations of such a curves, then the action functional does not necessarily vanish, and only those curves γ verifying the Euler–Lagrange equations are critical sections of the functional.

6. REGULARITY CONDITIONS

In the classical variational setting, a variational problem is said to be regular if there is a bijection between critical sections of the variational problem and solutions to the corresponding Hamilton's equations. The existence of such a bijection is assured by the fact that the Hessian matrix of the Lagrangian function is nondegenerate. We want to determine the regularity conditions in the graded variational setting. Developing the Euler–Lagrange equations associated to a Berezinian Lagrangian density, we obtain

$$(j^2\gamma)^*\left(x^h_{tt}\frac{\partial^2 L}{\partial x^h_t \partial x^i_t} + \cdots\right) = 0,$$

where the dots denote terms involving derivatives of order ≤ 1. In order to be able to write these equations down in the form

$$(j^2\gamma)^*\left(x^h_{tt}\right) = F_h\left(\gamma^*(x^i), (j^1\gamma)^*\left(x^h_t\right), (j^1\gamma)^*\left(x^h_s\right), (j^1\gamma)^*\left(x^h_{ts}\right)\right),$$

we must impose the matrix $(\partial^2 L/\partial x^h_t \partial x^i_t)$ to be nonsingular. In this case, the problem is said to be *regular*. Hence, the Euler–Lagrange equations of a regular Lagrangian are equivalent to a system of ordinary differential equations.

Proposition 6.1. *Let L be a homogeneous graded function on* $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$.

1. *If* $|L| = 0$*, then L is regular if*

$$\det\left(\partial^2 L/\partial x^h_t \partial x^i_t\right)^\sim \neq 0, \quad \text{for } h > 0,\ i > 0,$$

and

$$\det\left(\partial^2 L/\partial x_t^h \partial x_t^i\right)^{\sim} \neq 0, \quad \text{for } h < 0,\ i < 0.$$

2. *If* $|L| = 1$, *then* L *is regular if*

$$\det\left(\partial^2 L/\partial x_t^h \partial x_t^i\right)^{\sim} \neq 0, \quad \text{for } h > 0,\ i < 0.$$

Remark 6.2. Note that the second matrix in the even case is skew-symmetric, thus forcing n to be even, where $\dim(M, \mathcal{A}) = (m, n)$. Also note that in the odd case, $m = n$ necessarily.

7. POINCARÉ–CARTAN FORMS

Let us again recall that in the classical variational setting, once we have a regular variational problem, then there is also a bijection between the critical sections of the variational problem and the solutions to Hamilton's equation, which is defined in terms of the Poincaré–Cartan form. So, our next objective is to find a corresponding Poincaré–Cartan form in the graded setting. In Hernández Ruipérez and Muñoz Masqué (1984, Definition 2.8), a canonical graded 1-form—called the graded Poincaré–Cartan form—is associated to each first-order graded Lagrangian density λ. Here, we denote by $\Theta_0(\lambda)$ the graded Poincaré–Cartan form corresponding to $-\lambda$. If $\lambda = \mathrm{d}t \cdot f$, in local coordinates we have

$$\Theta_0(\lambda) = \left(\mathrm{d}x^i - \mathrm{d}t \cdot x_t^i - \mathrm{d}s \cdot x_s^i\right) \frac{\partial f}{\partial x_t^i} + \lambda,$$

where—as usual—we skip the index 0 in the indices running from negative to positive values. The forms $(\mathrm{d}x^i - \mathrm{d}t \cdot x_t^i - \mathrm{d}s \cdot x_s^i)$, $i = -n, \ldots, -1, 1, \ldots, m$, are called the standard contact forms on the 1-jet fibre bundle. Bearing the relationship between Berezinian variational problems and graded variational ones (Theorem 5.1) in mind, i.e.,

$$\lambda_\xi = \mathcal{L}_{\frac{d}{ds}}(\mathrm{d}t \cdot L),$$

it is natural to consider the graded 1-form

$$\Theta_\xi = \mathcal{L}_{\frac{d}{ds}} \Theta_0(\mathrm{d}t \cdot L)$$

as a Poincaré–Cartan form associated to the Berezinian Lagrangian density ξ.

First of all, note that from the very definition of Θ_ξ we obtain

$$\Theta_\xi = \left(\mathrm{d}x_s^i - \mathrm{d}t \cdot x_{st}^i\right) \frac{\partial L}{\partial x_t^i} + (-1)^{|x^i|}\left(\mathrm{d}x^i - \mathrm{d}t \cdot x_t^i - \mathrm{d}s \cdot x_s^i\right) \times \frac{d}{ds}\left(\frac{\partial L}{\partial x_t^i}\right) + \mathrm{d}t \cdot \frac{dL}{ds}. \tag{6}$$

Remark 7.1. The Poincaré–Cartan form thus defined is a well-defined differential form. Indeed, let $\{t', s'\}$ be another set of graded coordinates on $\mathbb{R}^{1|1}$. Then, there are real functions f, g such that both systems of graded coordinates are related by $t' = f(t), s' = g(t)s$, with $g(t) \neq 0,\ f'(t) \neq 0$, where f' denotes the derivative of f with respect to t. Accordingly, we have

$$\mathrm{d}t' = f'(t)\,\mathrm{d}t, \qquad \frac{d}{ds'} = \frac{1}{g}\frac{d}{ds},$$

and if a global section ξ of the first-order Berezinian sheaf has the following two expressions

$$\xi = \left[\mathrm{d}t \otimes \frac{d}{ds}\right] L = \left[\mathrm{d}t' \otimes \frac{d}{ds'}\right] L', \quad L, L' \in \mathcal{A}^1,$$

the relationship between these two functions is given by $L = (f'/g)L'$. The change of coordinates in $\mathbb{R}^{1|1}$ induces the following change of coordinates in J^2:

$$x^i_{t'} = x^i_t/f', \qquad x^i_{s'} = x^i_s/g, \qquad x^i_{s't'} = x^i_{st}/gf'.$$

According to this change of coordinates it is just a computation to check that the Poincaré–Cartan form is well defined. Indeed,

$$\begin{aligned}
\Theta'_\xi &= \mathcal{L}_{\frac{d}{ds'}}\left(\left(\mathrm{d}x^i - \mathrm{d}t' \cdot x^i_{t'} - \mathrm{d}s' \cdot x^i_{s'}\right)\frac{\partial L'}{\partial x^i_{t'}} + \mathrm{d}t' \cdot L'\right)\\
&= \mathcal{L}_{\frac{1}{g}\frac{d}{ds}}\left(\left(\mathrm{d}x^i - \mathrm{d}t \cdot x^i_t - \mathrm{d}s \cdot x^i_s\right)\frac{\partial((g/f')L)}{\partial x^i_{t'}} + \mathrm{d}t \cdot gL\right)\\
&= \mathcal{L}_{\frac{d}{ds}}\left(\left(\mathrm{d}x^i - \mathrm{d}t \cdot x^i_t - \mathrm{d}s \cdot x^i_s\right)\frac{\partial L}{\partial x^i_t} + \mathrm{d}t \cdot L\right) = \Theta_\xi.
\end{aligned}$$

7.1. The Sub-Bundle $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$

For a classical variational problem, the configuration space is a manifold M, and the Lagrangian is a function L on $J^1(\mathbb{R}, M) = \mathbb{R} \times TM$. The Poincaré–Cartan form is a 1-form on $\mathbb{R} \times TM$ (i.e., a section $\mathbb{R} \times TM \to T^*(\mathbb{R} \times TM)$), which projects onto a map $\mathbb{R} \times TM \to T^*(\mathbb{R} \times M)$. The analogous projection process can also be done in the graded setting. In principle, the Poincaré–Cartan form lives in the second order graded jet bundle but it should be noted that, from its local expression, Θ_ξ only depends on the coordinates $(t, s, x^i, x^i_t, x^i_s, x^i_{st})$; i.e., Θ_ξ does not depend on x^i_{tt}. Therefore, in order to develop a true Hamiltonian formalism, it would be desirable to project this form onto an appropriate fibre bundle. This is done below.

Let $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A})) \subset J^1(\mathbb{R}^{1|0}, J^1(\mathbb{R}^{0|1}, (M, \mathcal{A})))$ be the sub-bundle defined by $s_t = 0$. Then, there exists a canonical submersion over $\mathbb{R}^{1|1}$,

$$\pi\colon J^2(\mathbb{R}^{1|1}, (M, \mathcal{A})) \to J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$$

defined as follows: Each morphism $f\colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$ induces a family $f_t\colon \mathbb{R}^{0|1} \to (M, \mathcal{A})$, $t \in \mathbb{R}$, and, taking jets, $j^1(f_t)\colon \mathbb{R}^{0|1} \to J^1(\mathbb{R}^{0|1}, (M, \mathcal{A}))$. By composing $j^1(f_t)^*$ with the structure morphism $\Lambda^{\cdot}(\mathbb{R}) \to \mathbb{R}$ we obtain $[j^1(f_t)]^*\colon \mathcal{A}_{J^1(\mathbb{R}^{0|1},(M,\mathcal{A}))} \to \mathbb{R}$. Let

$$[j^1(f)]^*\colon \mathcal{A}_{J^1(\mathbb{R}^{0|1},(M,\mathcal{A}))} \to C^\infty(\mathbb{R})$$

be the ring homomorphism defined by

$$[j^1(f)]^*(a)\,(t) = [j^1(f_t)]^*(a),$$

and let

$$[j^1(f)]\colon \mathbb{R}^{0|1} \to J^1(\mathbb{R}^{0|1}, (M, \mathcal{A}))$$

be the corresponding morphism of graded manifolds. It is readily seen that the mapping $j^{1,1}(f) = j^1([j^1 f])$ takes values in $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, and also that

$$j^{1,1}\colon \mathrm{Mor}(\mathbb{R}^{1|1}, (M, \mathcal{A})) \to \Gamma(J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A})))$$

is a differential operator of second order. Consequently, $j^{1,1}$ must factor through $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, thus providing the desired submersion. From the previous local expression for Θ_ξ, we can conclude that Θ_ξ is π-projectable; its projection is also denoted by Θ_ξ.

Moreover, we remark that the role that $J^1(\mathbb{R}, M)$ plays in the classical case, is played by $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ in the graded case, but there is an outstanding difference: Whereas $J^1(\mathbb{R}, M)$ is equal to $\mathbb{R} \times TM$, this is no longer true in the graded case. The sub-bundle $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is not the product of $\mathbb{R}^{1|1}$ times the supertangent. Anyone of the different definitions of the supertangent bundle existing in the literature does not provide the right dimension. The graded dimension of this sub-bundle is $(1 + 2m + 2n, 1 + 2m + 2n)$ and the dimensions of the possible supertangent bundle are $(2m, 2n)$ (Kostant, 1977, 2.12) or $(2m + n, 2n + m)$ (Sánchez-Valenzuela, 1986).

8. HAMILTON EQUATIONS

We are now ready to state the equivalence between the Berezinian Lagrangian formalism and the Hamiltonian formalism in the graded case. Precisely,

Theorem 8.1. *A local section γ of the submersion $p_1\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ is a critical section for the Berezinian Lagrangian density ξ if and only if*

$$(j^2\gamma)^*(\iota_X \mathrm{d}\Theta_\xi) = 0, \tag{7}$$

for every vector field X on $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ vertical over $\mathbb{R}^{1|1}$.

Proof: Let us start by taking $X = \partial/\partial x_s^j$ for $j = -n, \dots, -1, 1, \dots, m$. Then, we have

$$\begin{aligned}
(j^2\gamma)^*\left(\iota_{\frac{\partial}{\partial x_s^j}} \mathrm{d}\Theta_\xi\right) &= (j^2\gamma)^*\left(-\mathrm{d}\left(\frac{\partial L}{\partial x_t^j}\right) + \mathrm{d}s \cdot \frac{d}{ds}\left(\frac{\partial L}{\partial x_t^j}\right) + \mathrm{d}t \cdot \frac{\partial}{\partial x_s^j}\left(\frac{dL}{ds}\right)\right) \\
&= (j^2\gamma)^*\left(-\mathrm{d}x^k \cdot \frac{\partial^2 L}{\partial x^k \partial x_t^j} - \mathrm{d}x_t^k \cdot \frac{\partial^2 L}{\partial x_t^k \partial x_t^j} - \mathrm{d}x_s^k \cdot \frac{\partial^2 L}{\partial x_s^k \partial x_t^j}\right. \\
&\quad + \mathrm{d}s \cdot x_s^k \frac{\partial^2 L}{\partial x^k \partial x_t^j} + \mathrm{d}s \cdot x_{ts}^k \frac{\partial^2 L}{\partial x_t^k \partial x_t^j} \\
&\quad \left. + \mathrm{d}t \cdot \left\{\left[\frac{\partial}{\partial x_s^j}, \frac{d}{ds}\right] L - (-1)^{|x^j|} \frac{d}{ds}\left(\frac{\partial L}{\partial x_s^j}\right)\right\}\right) \\
&= (j^2\gamma)^* \mathrm{d}t\left(\frac{\partial L}{\partial x^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_t^j}\right) - (-1)^{|x^j|}\frac{d}{ds}\left(\frac{\partial L}{\partial x_s^j}\right)\right),
\end{aligned}$$

where we have used that contact forms vanish when pulling them back along $j^2\gamma$ and also that $[\frac{\partial}{\partial x_s^j}, \frac{d}{ds}] = \frac{\partial}{\partial x^j}$. Next, taking $X = \partial/\partial x^j$, we have

$$\begin{aligned}
(j^2\gamma)^*\left(\iota_{\frac{\partial}{\partial x^j}} \mathrm{d}\Theta_\xi\right) &= (j^2\gamma)^*\left(-(-1)^{|x^j|}\mathrm{d}\left(\frac{d}{ds}\left(\frac{\partial L}{\partial x_t^j}\right)\right) + \mathrm{d}t \cdot \frac{\partial}{\partial x^j}\left(\frac{dL}{ds}\right)\right) \\
&= (-1)^{|x^j|}\mathcal{L}_{\frac{\partial}{\partial s}}(j^2\gamma)^*\left(-\mathrm{d}\left(\frac{\partial L}{\partial x_t^j}\right) + \mathrm{d}t \cdot \frac{\partial L}{\partial x^j}\right) \\
&= (-1)^{|x^j|}\mathcal{L}_{\frac{\partial}{\partial s}}(j^2\gamma)^*\left(-\mathrm{d}x^k \cdot \frac{\partial^2 L}{\partial x^k \partial x_t^j} - \mathrm{d}x_t^k \cdot \frac{\partial^2 L}{\partial x_t^k \partial x_t^j}\right. \\
&\quad \left. - \mathrm{d}x_s^k \cdot \frac{\partial^2 L}{\partial x_s^k \partial x_t^j} + \mathrm{d}t \cdot \frac{\partial L}{\partial x^j}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{|x^j|} \mathcal{L}_{\frac{\partial}{\partial s}} (j^2\gamma)^* \left(-\mathrm{d}t \cdot \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial x_t^j} \right) + \left(\frac{\partial L}{\partial x^j} \right) \right\} \right. \\
&\quad \left. + \ \mathrm{d}s \cdot \frac{d}{ds} \left(\frac{\partial L}{\partial x_t^j} \right) \right) \\
&= (-1)^{|x^j|} (j^2\gamma)^* \left(-\mathrm{d}t \cdot \frac{d}{ds} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial x_t^j} \right) \right. \right. \\
&\quad \left. \left. - \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial L}{\partial x_s^j} \right) + \ \frac{\partial L}{\partial x^j} \right\} \right).
\end{aligned}
$$

Finally, taking $X = \partial/\partial x_t^j$ or $X = \partial/\partial x_{ts}^j$ we deduce that $(j^2\gamma)^*(\iota_X \mathrm{d}\Theta_\xi)$ directly vanishes.

Therefore, for every vertical vector field X, we have

$$
(j^2\gamma)^*(\iota_X \mathrm{d}\Theta_\xi) = (j^2\gamma)^* \big(X \left(x_s^i \right) \Omega_i + (-1)^{|x^i|} X(x^i) \big(\mathcal{L}_{\frac{d}{ds}} \Omega_i \big) \big), \tag{8}
$$

where

$$
\Omega_i = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial x_t^i} \right) - (-1)^{|x^i|} \frac{d}{ds} \left(\frac{\partial L}{\partial x_s^i} \right) \right) \mathrm{d}t.
$$

□

In the previous theorem, the Eq. (7) is called the *Hamilton equation* for the Berezinian Lagrangian density ξ.

The following result states that—as in the ungraded case—holonomy of the solutions to Hamilton equations is a consequence of regularity.

Theorem 8.2. *Let Θ_ξ be the graded 1-form associated to the Berezinian Lagrangian density $\xi = [\mathrm{d}t \otimes \frac{d}{ds}]L$. We have*

(i) *If γ is a Berezinian critical section, then $(j^{1,1}\gamma)^*(\iota_X \mathrm{d}\Theta_\xi) = 0$ for every vector field X on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ vertical over $\mathbb{R}^{1|1}$.*

(ii) *Conversely, assume L is regular and that $\overline{\gamma}\colon \mathbb{R}^{1|1} \to J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is a section such that, $\overline{\gamma}^*(\iota_X \mathrm{d}\Theta_\xi) = 0$ for every vector field X on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ vertical over $\mathbb{R}^{1|1}$. Then, there exists a unique Berezinian critical section γ such that $\overline{\gamma} = j^{1,1}(\gamma)$.*

Proof: The first part of the statement follows taking into account that formula (8) in the proof of the previous theorem remains true for $j^{1,1}\gamma$. For the second part, we just use the following equations and the fact that L is regular:

$$\gamma^*\left(\iota_{\frac{\partial}{\partial x^h_{st}}}\mathrm{d}\Theta_\xi\right) = (-1)^{|x^h||x^i|}\overline{\gamma}^*\left(\left(\mathrm{d}x^i - \mathrm{d}t\cdot x^i_t - \mathrm{d}s\cdot x^i_s\right)\frac{\partial^2 L}{\partial x^h_t\partial x^i_t}\right) = 0,$$

$$\gamma^*\left(\iota_{\frac{\partial}{\partial x^h_{t}}}\mathrm{d}\Theta_\xi\right) = (-1)^{|x^h|(1+|x^i|)}\overline{\gamma}^*\left(\left(\mathrm{d}x^i_s - \mathrm{d}t\cdot x^i_{st}\right)\frac{\partial^2 L}{\partial x^h_t\partial x^i_t}\right.$$
$$\left. + (-1)^{|x^i|}\left(\mathrm{d}x^i - \mathrm{d}t\cdot x^i_t - \mathrm{d}s\cdot x^i_s\right)\frac{d}{ds}\left(\frac{\partial^2 L}{\partial x^h_t\partial x^i_t}\right)\right) = 0. \quad \square$$

9. CANONICAL COORDINATES

From now on we assume that the Lagrangian function under consideration is homogeneous.

The following useful result involving regular Lagrangians allows us to construct a set of graded coordinates on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ adapted to the variational problem.

Proposition 9.1. *Let L be a regular homogeneous Lagrangian function, then the set of graded functions*

$$\left(t, s, x^i, x^i_s, p_i = \frac{\partial L}{\partial x^i_t}, p_{i,s} = \frac{d}{ds}\left(\frac{\partial L}{\partial x^i_t}\right)\right) \tag{9}$$

is a system of graded coordinates on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$.

Remark 9.2. The degrees of the new coordinates are $|p_i| = |L| + |x^i|$, $|p_{i,s}| = |L| + |x^i| + 1$.

Proof: It is easy to check that the graded Jacobian of the set of functions $(t, s, x^i, x^i_s, p_i, p_{i,s})$ with respect to the graded coordinates $(t, s, x^i, x^i_t, x^i_s, x^i_{st})$ is, up to a sign, equal to $\det(\partial^2 L/\partial x^h_t \partial x^i_t)$. Thus, L is regular if and only if the Jacobian is invertible; i.e., if and only if

$$\det\left(\frac{\partial^2 L}{\partial x^h_t\partial x^i_t}\right)^{\sim} \neq 0.$$

This condition is equivalent to the regularity of the homogeneous function L. $\square$

Let us now write the graded Poincaré–Cartan form (6) of a regular homogeneous Lagrangian L in terms of the canonical coordinates (9). We have

$$\begin{aligned}\Theta_\xi &= \left(\mathrm{d}x^i_s - \mathrm{d}t\cdot x^i_{st}\right)p_i + (-1)^{|x^i|}\left(\mathrm{d}x^i - \mathrm{d}t\cdot x^i_t - \mathrm{d}s\cdot x^i_s\right)p_{i,s} + \frac{dL}{ds}\,\mathrm{d}t \\ &= \mathrm{d}x^i_s\cdot p_i + (-1)^{|x^i|}\mathrm{d}x^i\cdot p_{i,s} + \mathrm{d}t H_t + \mathrm{d}s\cdot H_s,\end{aligned}$$

where

$$H_t = -x^i_{st} p_i - (-1)^{|x^i|} x^i_t p_{i,s} + \frac{dL}{ds} = \frac{d}{ds}\left(-x^i_t p_i + L\right),$$
$$H_s = (-1)^{1+|x^i|} x^i_s p_{i,s} = \frac{d}{ds}\left(x^i_s p_i\right). \tag{10}$$

These two functions, H_t and H_s, play the role of Hamiltonian functions, so they deserve to be called the t-Hamiltonian and s-Hamiltonian respectively. Note that $|H_t| = |L| + 1$, $|H_s| = |L|$.

10. THE RADICAL OF THE POINCARÉ–CARTAN FORM

Below we compute the radical of the exterior differential of the Poincaré–Cartan form

$$\mathrm{d}\Theta_\xi = -\mathrm{d}x^i_s \wedge \mathrm{d}p_i - (-1)^{|x^i|}\mathrm{d}x^i \wedge \mathrm{d}p_{i,s} - \mathrm{d}t \wedge \mathrm{d}H_t - \mathrm{d}s \wedge \mathrm{d}H_s.$$

Note that $\mathrm{d}\Theta_\xi$ is a 2-form whose degree depends on the degree of L; in fact, $|\Theta_\xi| = |L| + 1$.

If $X = X_0 + X_1$ is an arbitrary graded vector field, then

$$\begin{aligned}
\iota_X \mathrm{d}\Theta_\xi = &- X_0\left(x^i_s\right)\mathrm{d}p_i + \mathrm{d}x^i_s \cdot X_0(p_i) - X_1\left(x^i_s\right)\mathrm{d}p_i - (-1)^{|x^i|}\mathrm{d}x^i_s \cdot X_1(p_i)\\
&- (-1)^{|x^i|} X_0(x^i)\,\mathrm{d}p_{i,s} + (-1)^{|x^i|}\mathrm{d}x^i \cdot X_0(p_{i,s})\\
&- (-1)^{|x^i|} X_1(x^i)\,\mathrm{d}p_{i,s} + \mathrm{d}x^i \cdot X_1(p_{i,s})\\
&- X_0(t)\,\mathrm{d}H_t + \mathrm{d}t \cdot X_0(H_t) - X_1(t)\,\mathrm{d}H_t + \mathrm{d}t \cdot X_1(H_t)\\
&- X_0(s)\,\mathrm{d}H_s + \mathrm{d}s \cdot X_0(H_s) - X_1(s)\,\mathrm{d}H_s - \mathrm{d}s \cdot X_1(H_s).
\end{aligned}$$

The equation $\iota_X \mathrm{d}\Theta_\xi = 0$ can be splitted into six graded equations according to the coefficient of the differentials of the graded coordinates. Again these equations can be splitted according to their parity. Thus the coefficients of dx^i, dp_i, dx^i_s, $dp_{i,s}$ give rise to the following eight equations:

$$X_1\left(x^i_s\right) + X_1(t)\frac{\partial H_t}{\partial p_i} = 0, \qquad X_0\left(x^i_s\right) + X_0(t)\frac{\partial H_t}{\partial p_i} = 0,$$

$$X_0(p_i) - X_0(t)\frac{\partial H_t}{\partial x^i_s} - X_0(s)p_{i,s} = 0,$$

$$X_1(p_i) - (-1)^{|x^i|} X_1(t)\frac{\partial H_t}{\partial x^i_s} - X_1(s)p_{i,s} = 0,$$

$$X_0(x^i) + (-1)^{|x^i|} X_0(t) \frac{\partial H_t}{\partial p_{i,s}} - X_0(s) x^i_s = 0,$$

$$X_1(x^i) + (-1)^{|x^i|} X_1(t) \frac{\partial H_t}{\partial p_{i,s}} - X_1(s) x^i_s = 0,$$

$$X_1(p_{i,s}) - (-1)^{|x^i|} X_1(t) \frac{\partial H_t}{\partial x^i} = 0, \qquad X_0(p_{i,s}) - (-1)^{|x^i|} X_0(t) \frac{\partial H_t}{\partial x^i} = 0. \quad (11)$$

The consequence of this set of equations is the following:

Theorem 10.1. *The radical of* $\mathrm{d}\Theta_\xi$ *is a free module of rank 2 and of type* $(1, 1)$, *with free basis the pair of graded vector fields* X^t, X^s *uniquely determined by the conditions*

$$X^t(t) = 1, \qquad X^t(s) = 0, \quad |X^t| = 0,$$

$$X^s(t) = 0, \qquad X^s(s) = 1, \quad |X^s| = 1.$$

Moreover the radical is involutive and $[X^t, X^s] = [X^s, X^s] = 0$.

Remark 10.2. The two commuting relations satisfied by X^t and X^s above, are precisely the required conditions for the flow Γ of $X^t + X^s$ to satisfy (2). This means that $X^t + X^s$ is a distinguished vector field from the point of view of the integration problem (see Section 3.1). Equations (11) together with conditions $\iota_{X^t+X^s}\mathrm{d}\,\Theta_\xi = 0$ are called the graded Euler–Lagrange equations of the Berezinian variational problem defined by ξ.

Finally, the odd and even parts of the coefficients of dt and ds in $\iota_X d\Theta_\xi = 0$, can be written, for a graded vector field satisfying the previous four, as

$$X_0(s)\frac{dH_t}{ds} = 0, \qquad X_1(s)\frac{dH_t}{ds} = 0,$$
$$X_1(t)\frac{dH_t}{ds} = 0, \qquad X_0(t)\frac{dH_t}{ds} = 0. \quad (12)$$

11. HAMILTON'S EQUATIONS IN CANONICAL COORDINATES

Next, let us come back to the solutions of Hamilton's equations. We want to determine those curves $\bar{\gamma}\colon \mathbb{R}^{1|1} \to J^{1,1}$ such that $\bar{\gamma}^*(\iota_X \mathrm{d}\Theta_\xi) = 0$ for all $X \in \mathcal{X}(J^{1,1})$ vertical over $\mathbb{R}^{1|1}$.

Theorem 11.1. *Let* $\xi = [\mathrm{d}t \otimes \frac{d}{ds}]L$ *be a regular homogeneous Berezinian Lagrangian density on* $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$. *The map* $\gamma \mapsto j^{1,1}(\gamma)$, *states a bijection between the set of critical sections of* ξ *and the set of curves* $\bar{\gamma}\colon \mathbb{R}^{1|1} \to$

$J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ *that satisfy the following system of six differential equations:*

$$(-1)^{|x^i|}\frac{\partial(\overline{\gamma}^* p_{i,s})}{\partial t} = \overline{\gamma}^*\left(\frac{\partial H_t}{\partial x^i}\right), \qquad \frac{\partial(\overline{\gamma}^* p_{i,s})}{\partial s} = 0,$$
$$\frac{\partial(\overline{\gamma}^* p_i)}{\partial t} = \overline{\gamma}^*\left(\frac{\partial H_t}{\partial x^i_s}\right),$$
$$-\frac{\partial\left(\overline{\gamma}^* x^j_s\right)}{\partial t} = \overline{\gamma}^*\left(\frac{\partial H_t}{\partial p_i}\right), \qquad \frac{\partial\left(\overline{\gamma}^* x^i_s\right)}{\partial s} = 0,$$
$$-(-1)^{|x^i|}\frac{\partial\left(\overline{\gamma}^* x^i\right)}{\partial t} = \overline{\gamma}^*\left(\frac{\partial H_t}{\partial p_{i,s}}\right). \tag{13}$$

The differential equations have been written in canonical coordinates.

Proof: First, note

$$\iota_{\frac{\partial}{\partial x^i}}\mathrm{d}\Theta_\xi = -(-1)^{|x^i|}\mathrm{d}p_{i,s} + \mathrm{d}t\frac{\partial H_t}{\partial x^i} + (-1)^{|x^i|}\mathrm{d}s\frac{\partial H_s}{\partial x^i}.$$

We know that $\overline{\gamma}^*(t) = t$, $\overline{\gamma}^*(s) = s$ as $\overline{\gamma} = j^{1,1}(\gamma)$ for some curve $\gamma\colon \mathbb{R}^{1|1} \to (M, \mathcal{A})$, because L is regular (Theorem 8.2-(ii)). Moreover, the equation $\overline{\gamma}^*(\iota_{\frac{\partial}{\partial x^i}}\mathrm{d}\Theta_\xi) = 0$ can be written as

$$(-1)^{|x^i|}\,\mathrm{d}(\overline{\gamma}^* p_{i,s}) = \mathrm{d}t\cdot\overline{\gamma}^*\left(\frac{\partial H_t}{\partial x^i}\right) + (-1)^{|x^i|}\mathrm{d}s\cdot\overline{\gamma}^*\left(\frac{\partial H_s}{\partial x^i}\right),$$

and this equation is equivalent to the following two

$$(-1)^{|x^i|}\frac{\partial(\overline{\gamma}^* p_{i,s})}{\partial t} = \overline{\gamma}^*\left(\frac{\partial H_t}{\partial x^i}\right), \qquad \frac{\partial(\overline{\gamma}^* p_{i,s})}{\partial s} = 0,$$

where the definition of the function H_s has been recalled. The same procedure with the rest of the vertical coordinate vector fields gives the following equations:

$$\begin{aligned}
\frac{\partial(\overline{\gamma}^* p_i)}{\partial t} &= \overline{\gamma}^*\left(\frac{\partial H_t}{\partial x^i_s}\right), & \frac{\partial(\overline{\gamma}^* p_i)}{\partial s} &= \overline{\gamma}^*(p_{i,s}),\\
\frac{\partial\left(\overline{\gamma}^* x^i_s\right)}{\partial t} &= -\overline{\gamma}^*\left(\frac{\partial H_t}{\partial p_i}\right), & \frac{\partial\left(\overline{\gamma}^* x^i_s\right)}{\partial s} &= 0,\\
\frac{\partial(\overline{\gamma}^* x^i)}{\partial t} &= -(-1)^{|x^i|}\overline{\gamma}^*\left(\frac{\partial H_t}{\partial p_{i,s}}\right), & \frac{\partial(\overline{\gamma}^* x^i)}{\partial s} &= \overline{\gamma}^*\left(x^i_s\right).
\end{aligned}$$

Two of these equations are tautological as $\overline{\gamma} = j^{1,1}(\gamma)$. Thus, only six equations are relevant. □

12. SOLVING THE HAMILTON EQUATIONS

In the classical setting, it is well known that the solutions to Hamilton equations are uniquely defined by the points in the tangent bundle; i.e., by tangent vectors. Each tangent vector acts as an initial condition for the Hamilton equations, or, equivalently, as the initial point of an integral curve of the Euler vector field.

Now, in order to solve the Hamilton equations it is necessary to introduce the notion of an initial condition in this setting. Precisely, an *initial condition* for the system of equations in Theorem 11.1 is a graded $\mathbb{R}$-algebra homomorphism $\chi\colon \mathcal{A}(J^{1,1}) \to \Lambda^{\cdot}\mathbb{R}$ such that $\chi(s) = s$, $\chi(t) = 0$. Let $M^{1,1}$ be the underlying manifold to $J^{1,1}$. A homomorphism χ is said to be over the point $z \in M^{1,1}$ if the composition map

$$\mathcal{A}(J^{1,1}) \xrightarrow{\chi} \Lambda^{\cdot}\mathbb{R} \xrightarrow{\sim} \mathbb{R}$$

coincides with the linear form $ev_z(f) = \tilde{f}(z)$. In this case, we write $\chi = \chi_z$. A curve $\overline{\gamma}\colon \mathbb{R}^{1|1} \to J^{1,1}$ is said to satisfy the initial condition χ if the composition

$$\mathcal{A}(J^{1,1}) \xrightarrow{\overline{\gamma}^*} \mathcal{A}(\mathbb{R}^{1,1}) \xrightarrow{\overline{ev}_{(t=0)}} \Lambda^{\cdot}\mathbb{R},$$

where $\overline{ev}_{(t=0)}(f(t) + g(t)s) = f(0) + g(0)s$, coincides with χ.

Theorem 12.1. *Given an initial condition χ, there is a unique solution $\overline{\gamma}$ of the Eqs.* (13) *that satisfies the initial condition χ.*

The existence of the Euler vector field in this setting (i.e., a vector field whose integral curves are in one-to-one correspondence with the extremals of the variational problem) is assured by the following

Theorem 12.2. *Let L be a regular homogenous Lagrangian and let X be the graded vector field belonging to the radical of $\mathrm{d}\,\Theta_\xi$ defined by the conditions $X(t) = 1$, $X(s) = 1$ (i.e., $X = X^t + X^s$). Given an initial condition $\chi\colon \mathcal{A}(J^{1,1}) \to \Lambda^{\cdot}\mathbb{R}$, let $\overline{\chi}\colon \mathcal{A}(\mathbb{R}^{1|1} \times J^{1,1}) \to \mathcal{A}(\mathbb{R}^{1|1})$ be the composition*

$$\mathcal{A}(\mathbb{R}^{1|1} \times J^{1,1}) \xrightarrow{\mathrm{id}\otimes\chi} \mathcal{R}^{1|1} \otimes \Lambda^{\cdot}\mathbb{R} \xrightarrow{ev_{(s'=0)}} C^\infty(\mathbb{R}) \otimes \Lambda^{\cdot}\mathbb{R} = \mathcal{A}(\mathbb{R}^{1|1}).$$

Let Γ be the integral flow of X and let $\hat{\gamma}\colon \mathbb{R}^{1|1} \to J^{1,1}$ be the curve defined by the condition $\hat{\gamma}^* = \overline{\chi} \circ \Gamma^*$. Then, $\hat{\gamma}$ is the unique solution to the differential equations (13) *that satisfies the initial condition $\chi' = \overline{ev}_{(t=0)} \circ \hat{\gamma}^*$. Moreover $\chi = \chi'$, thus, X determines the whole set of solutions to the variational problem defined by the Lagrangian L.*

Proof: We first remark that χ' is an initial condition as we have

$$\chi'(s) = (\overline{ev}_{(t=0)} \circ \hat{\gamma}^*)(s) = (\overline{ev}_{(t=0)} \circ \overline{\chi} \circ \Gamma^*)(s)$$
$$= (\chi \circ ev_{(t'=0)} \circ \Gamma^*)(s) = \chi(s) = s.$$

Similarly, $\chi'(t) = 0$. Since $ev_{(t'=0)} \circ \Gamma^* = \mathrm{id}$, we conclude $\chi' = \chi$. We have only to prove that $\hat{\gamma}$ is a solution to the Eqs. (13). Indeed,

$$\hat{\gamma}^*(X(x^i)) = (-1)^{1+|x^i|}\hat{\gamma}^*\left(\frac{\partial H_t}{\partial p_{i,s}} + x_s^i\right)$$
$$= (\overline{\chi} \circ \Gamma^* \circ X)(x^i) = \left(\overline{\chi} \circ \left(\frac{\partial}{\partial t'} + \frac{\partial}{\partial s'}\right) \circ \Gamma^*\right)(x^i)$$
$$= \left(\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \circ \hat{\gamma}^*\right)(x^i) = \frac{\partial \hat{\gamma}^*(x^i)}{\partial t} + \frac{\partial \hat{\gamma}^*(x^i)}{\partial s}.$$

Equating now the homogeneous parts, we obtain

$$\frac{\partial \hat{\gamma}^*(x^i)}{\partial t} = -(-1)^{|x^i|}\hat{\gamma}^*\left(\frac{\partial H_t}{\partial p_{i,s}}\right), \qquad \frac{\partial \hat{\gamma}^*(x^i)}{\partial s} = -(-1)^{|x^i|}\hat{\gamma}^*\left(x_s^i\right).$$

The first set of equations are exactly the equations in (13) and, the second set is tautological. Hence from the conditions

$$\hat{\gamma}^*\left(X\left(x_s^i\right)\right) = -\hat{\gamma}^*\left(\frac{\partial H_t}{\partial p_i}\right) = \frac{\partial \hat{\gamma}^*\left(x_s^i\right)}{\partial t} + \frac{\partial \hat{\gamma}^*\left(x_s^i\right)}{\partial s},$$

we obtain

$$\frac{\partial \hat{\gamma}^*\left(x_s^i\right)}{\partial t} = -\hat{\gamma}^*\left(\frac{\partial H_t}{\partial p_i}\right), \qquad \frac{\partial \hat{\gamma}^*\left(x_s^i\right)}{\partial s} = 0.$$

These equations are exactly two groups of equations in (13). Also, from

$$\hat{\gamma}^*(X(p_i)) = \hat{\gamma}^*\left(\frac{\partial H_t}{\partial x_s^i} + p_{i,s}\right),$$

we obtain

$$\frac{\partial \hat{\gamma}^*(p_i)}{\partial t} = \hat{\gamma}^*\left(\frac{\partial H_t}{\partial x_s^i}\right), \qquad \frac{\partial \hat{\gamma}^*(p_i)}{\partial s} = \hat{\gamma}^*(p_{i,s}).$$

The first set of equations is exactly another group of equations in (13) and the second set is tautological. Finally, from

$$\hat{\gamma}^*(X(p_{i,s})) = (-1)^{|x^i|}\hat{\gamma}^*\left(\frac{\partial H_t}{\partial x^i}\right),$$

we obtain

$$\frac{\partial \hat{\gamma}^*(p_{i,s})}{\partial t} = (-1)^{|x^i|}\hat{\gamma}^*\left(\frac{\partial H_t}{\partial x^i}\right), \qquad \frac{\partial \hat{\gamma}^*(p_{i,s})}{\partial s} = 0.$$

These equations are exactly the two first groups of equations in (13). $\square$

Remark 12.3. The vector field of motion $X = X^t + X^s$, also called the Euler vector field, deserves some words. This vector field is defined by the following conditions: (i) its even part, X^t, projects onto $\partial/\partial t$, (ii) its odd part, X^s, projects onto $\partial/\partial s$, and (iii) X^t, X^s is a basis of the radical of the differential of the Poincaré–Cartan form.

Moreover, if Γ denotes the integral flow of $X^t + X^s$ then, due to the commutation relations $[X^t, X^s] = [X^s, X^s] = 0$, the Euler vector field and the integration model $\partial/\partial t + \partial/\partial s$ are Γ-related (see the Eq. (2) and Remark 10.2).

Its final local expression is the following:

$$X = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \left(x^i_s - (-1)^{|x^i|}\frac{\partial H_t}{\partial p_{i,s}}\right)\frac{\partial}{\partial x^i} - \frac{\partial H_t}{\partial p_i}\frac{\partial}{\partial x^i_s}$$
$$+ \left(p_{i,s} + \frac{\partial H_t}{\partial x^i_s}\right)\frac{\partial}{\partial p_i} + (-1)^{|x^i|}\frac{\partial H_t}{\partial x^i}\frac{\partial}{\partial p_{i,s}}.$$

As in the classical case, we have shown that its flow determines all the solutions of the regular variational problem. Moreover, it allows to put additional structures on the set of solutions. This is what we do in the next section.

13. GRADED AND SYMPLECTIC STRUCTURES ON THE MANIFOLD OF SOLUTIONS

The next result states that the set of solutions of Hamilton's equations can be endowed with a structure of graded manifold. Moreover, the differential of the Poincaré–Cartan form can be projected to this graded manifold and its projection is a graded symplectic form on it.

Theorem 13.1 (*The manifold of solutions*). *Let L be a regular homogeneous Lagrangian and let $\mathcal{A}_S$ be the ring of first integrals of the distribution defined by the radical of* $\mathrm{d}\,\Theta_\xi$ *(i.e., the first integrals common to both X^t and X^s). Then,*

1. $\mathrm{d}\,\Theta_\xi$ *can be irreducibly projected onto $\mathcal{A}_S$, thus defining a symplectic structure of degree $|L| + 1$.*
2. *The $\mathbb{R}$-graded algebras homomorphisms $\chi^S\colon \mathcal{A}_S \to \Lambda^{\cdot}\mathbb{R}$ can be identified with the initial conditions of the differential equations* (13).

Proof: The first statement follows from the previous results. To prove the second one, let us note that $\mathcal{A}_S$ is a subring of $\mathcal{A}(J^{1,1})$. Then, it is clear that each $\chi\colon \mathcal{A}(J^{1,1}) \to \Lambda^{\cdot}\mathbb{R}$ induces, by restriction, a homomorphism $\chi^S\colon \mathcal{A}_S \to \Lambda^{\cdot}\mathbb{R}$. The result now follows bearing in mind that, locally, $\mathcal{A}(J^{1,1}) = \mathcal{R}^{1|1}\hat{\otimes}\mathcal{A}_S$, where $\mathcal{A}\hat{\otimes}\mathcal{B}$ stands for the completion of $\mathcal{A}\otimes\mathcal{B}$ with respect to Grothendieck's π-topology. For the details see (Hernández Ruipérez and Muñoz Masqué, 1984, I, Theorem 3). □

Remark 13.2 (Local expression of the symplectic form). If L is a regular homogeneous Lagrangian, then the set of graded functions $(x^i, x^i_s, p_i, p_{i,s})$, $i = -n, \ldots, -1, 1, \ldots, m$, where p_i, $p_{i,s}$ are defined in (9), is a system of graded coordinates on the graded manifold of solutions and the expression of the symplectic form in this set of canonical coordinates is

$$-\mathrm{d}x^i_s \wedge \mathrm{d}p_i - (-1)^{|x^i|}\mathrm{d}x^i \wedge \mathrm{d}p_{i,s} = -\frac{d}{ds}(\mathrm{d}x^i \wedge \mathrm{d}p_i).$$

Remark 13.3. If $\dim(M, \mathcal{A}) = (m, n)$, then $\dim S = (2(m+n), 2(m+n))$. The dimension of this graded manifold allows both cases of homogeneous symplectic graded manifolds: It allows odd symplectic forms because the dimension, $2(m+n)$, of the underlying manifold agrees with the dimension of the graded ring and it allows even symplectic forms because the dimension of the underlying manifold is even (see Monterde, 1992b).

Remark 13.4. In spite of the fact that in the ungraded case the manifold of solutions is the tangent bundle, this is no longer true in the graded case. The graded manifold of solutions is not the supertangent bundle of $(M, \mathcal{A})$. See the last paragraph of Section 7.1.

Remark 13.5. Another remarkable fact is the change of parity. If the initial graded Lagrangian is even (resp. odd), then the resulting symplectic form is odd (resp. even). This means that if one wants to generalize a classical Lagrangian, then it is natural to choose an even Lagrangian whose image through the natural map be the classical Lagrangian. Then, the simplectic form will be an odd symplectic form. Let us recall that odd symplectic forms are simpler than the even ones. It is well known that in order to define an even symplectic form we need, at least, a classical symplectic form and a nondegenerate metric structure on some trivializing bundle. In order to define an odd simplectic form, however, just a bundle isomorphism between the tangent bundle of the underlying manifold and the trivializing bundle is needed.

14. COMPARISON WITH THE KOSZUL–SCHOUTEN BRACKETS

A classical particle is a curve $\gamma\colon \mathbb{R} \to M$ taking values in a differentiable manifold M. The generalization of a classical partical is the following: A particle is a curve $\gamma\colon \mathbb{R}^{1|1} \to (M, C_M^\infty)$. Note that a curve $\gamma\colon \mathbb{R}^{1|1} \to (M, C_M^\infty)$ is totally determined by its footprint on the base manifolds $\tilde{\gamma}\colon \mathbb{R} \to M$. The difference with the classical notion is that we allow not only $\mathbb{R}$-variations, but also $\mathbb{R}^{1|1}$-variations. This means that we work with maps $\Gamma\colon \mathbb{R}^{2|2} \to (M, C_M^\infty)$, or equivalently, with maps $\Gamma^*\colon C_M^\infty \to \mathcal{R}^{2|2}$. Thus, the image of a coordinate function x^i can be written as follows: $\Gamma^*(x^i) = f^i(t, \overline{t}) + h^i(t, \overline{t})s\overline{s}$, $i = -n, \ldots, -1, 1, \ldots, m$. Note that we introduce $\mathbb{R}^{2|2}$ by categorical arguments. First, we substitute $\mathbb{R}$-curves on a manifold by $\mathbb{R}^{1|1}$-curves on the graded manifold (M, C_M^∞), and nothing is changed. Both sets of curves are the same. Secondly, we use $\mathbb{R}^{1|1}$-variations of $\mathbb{R}^{1|1}$-curves, but this is equivalent to study maps from $\mathbb{R}^{2|2} \to (M, C_M^\infty)$. Some authors introduce $\mathbb{R}^{2|2}$ by other arguments. For example in Freed (1999, p. 22), the reason to introduce two odd variables is just to obtain a nonvanishing evaluation of a particular Lagrangian density.

All along this section, we study classical regular Lagrangians. In a certain sense, we can say that we look at classical problems from a graded point of view.

Let us consider the following particular case of graded manifold: (M, C_M^∞), i.e., a graded manifold with no odd coordinates; its graded dimension is $(n, 0)$, $n = \dim M$. A system of graded coordinates for this graded manifold is of the kind (x^i), $i = 1, \ldots, n$, with no coordinates of negative indices. Let us also consider a classical regular Lagrangian $L \in C^\infty(J^1(p_1\colon \mathbb{R} \times M \to \mathbb{R}))$. This function can be lifted to $J^1(\mathbb{R}^{1|1}, (M, C^\infty(M)))$ and then, we can build up the first order Berezinian density $[\mathrm{d}t \otimes \frac{d}{ds}]L$. Note that, due to the absence of odd coordinates in (M, C_M^∞), the Lagrangian L is also regular from the point of view of graded variational calculus.

Applying the previous deductions we obtain

1. A graded manifold of solutions of graded dimension $(2n, 2n)$, whose canonical coordinates are $(x^i, p_i, x_s^i, p_{i,s})$, $i = 1, \ldots, n$, and
2. An odd symplectic form that in canonical coordinates is written as

$$\omega_S = -\mathcal{L}_{\frac{d}{ds}}(\mathrm{d}x^i \wedge \mathrm{d}p_i) = -\mathrm{d}x_s^i \wedge \mathrm{d}p_i - \mathrm{d}x^i \wedge \mathrm{d}p_{i,s}.$$

Example 14.1. Let us consider the concrete example of the following classical regular Lagrangian on $M = \mathbb{R}^n$: $L = \frac{1}{2}\sum_{i=1}^n (x_t^i)^2$. According to Example 5.4, its graded Euler–Lagrange equations are simply $j^2(\gamma)^* x_{tt}^i = 0$ for $i = 1, \ldots, n$. This agrees with the aforementioned fact that curves $\gamma\colon \mathbb{R}^{1|1} \to (M, C^\infty(M))$ are totally determined by their footprint on the base manifolds $\tilde{\gamma}\colon \mathbb{R} \to M$. Thus, in this case, a graded curve γ, is a solution to the graded Euler–Lagrange equations if and only if $\tilde{\gamma}$ is a solution to the classical Euler–Lagrange equations. The canonical coordinates are: $p_i = x_t^i$, $p_{i,s} = x_{ts}^i$, $i = 1, \ldots, n$, and the odd symplectic form on the graded manifold of solutions is given by: $\omega_S = -\sum_{i=1}^n \mathrm{d}x_s^i \wedge \mathrm{d}x_t^i - \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}x_{ts}^i$.

14.1. Koszul–Schouten Bracket

Let (N, ω) be a symplectic manifold of dimension $2n$. As it is well known, one can construct an odd symplectic form on the graded manifold (N, Ω_N). This is the form whose odd Poisson bracket is the Koszul–Schouten bracket. This bracket is defined as follows: Let P be the associated nondegenerated Poisson bivector and let $\#_P\colon T^*N \to TN$ be the induced isomorphism. We denote the bracket by $[\![\,,\,]\!]_P\colon \Omega(N) \times \Omega(N) \to \Omega(N)$. Its action on differentiable 0 and 1-forms is given by

$$[\![f, g]\!]_P = 0, \qquad [\![f, dg]\!]_P = P(df, dg), \qquad [\![df, dg]\!]_P = d(P(df, dg)), \quad (14)$$

for any $f, g \in C^\infty(N)$, and it can be extended to the whole algebra of differentiable forms by using the Leibniz identity. See (Koszul, 1985) or (Khudaverdian, 1998) for the details. Let us take a system of Darboux coordinates (y^i, y^{i+n}), $i = 1, \ldots, n$, for (N, ω) such that $\omega = \sum_{i=1}^n dy^i \wedge dy^{i+n}$, and the associated Poisson tensor is written as $P = \sum_{i=1}^n (\partial/\partial y^i) \wedge (\partial/\partial y^{i+n})$. The associated system of graded coordinates on (N, Ω_N) is $(y^i, y^{i+n}, y^{-i}, y^{-i-n})$, $i = 1, \ldots, n$, where we recall that a coordinate with negative index, y^{-i}, denotes an odd generator, dy^i.

Lemma 14.2. *With this system of graded coordinates, the odd symplectic form associated to the odd Poisson bracket defined by Koszul can be written as*

$$\omega_K = \sum_{i=1}^n \mathrm{d}y^{-i} \wedge \mathrm{d}y^{i+n} + \mathrm{d}y^i \wedge \mathrm{d}y^{-i-n}.$$

Proof: The Hamiltonian vector fields defined by the graded coordinate functions are (see Beltrán and Monterde, 1995, Proposition 2.6)

$$[\![y^i, \cdot]\!] = -\iota_{\frac{\partial}{\partial y^{i+n}}}, \quad [\![y^{i+n}, \cdot]\!] = \iota_{\frac{\partial}{\partial y^i}}, \quad [\![\mathrm{d}y^i, \cdot]\!] = -\mathcal{L}_{\frac{\partial}{\partial y^{i+n}}}, \quad [\![\mathrm{d}y^{i+n}, \cdot]\!] = \mathcal{L}_{\frac{\partial}{\partial y^i}}.$$

Recall that the odd symplectic form, ω, associated to a nondegenerate odd Poisson bracket, $[\![\cdot, \cdot]\!]$, is defined by the formula $[\![\alpha, \beta]\!] = -\langle D_\alpha, D_\beta; \omega\rangle$, where D_α denotes the Hamiltonian vector field associated to $\alpha \in \Omega(M)$.

Let us compute the associated odd symplectic form, ω_K, for the Koszul–Schouten bracket. First, remark that for the graded coordinate system $(y^i, y^{i+n}, y^{-i}, y^{-i-n})$, $i = 1, \ldots, n$, the associated local basis of graded vector fields is

$$\left(\mathcal{L}_{\frac{\partial}{\partial y^i}}, \mathcal{L}_{\frac{\partial}{\partial y^{i+n}}}, \iota_{\frac{\partial}{\partial y^i}}, \iota_{\frac{\partial}{\partial y^{i+n}}}\right), \quad i = 1, \ldots, n,$$

and the associated local basis of graded 1-forms is $(\mathrm{d}y^i, \mathrm{d}y^{i+n}, \mathrm{d}y^{-i}, \mathrm{d}y^{-i-n})$, for $i = 1, \ldots, n$. Accordingly,

$$\left\langle \mathcal{L}_{\frac{\partial}{\partial y_{i+n}}}, \iota_{\frac{\partial}{\partial y^j}}; \omega_K \right\rangle = -[\![\mathrm{d}y^i, y^{j+n}]\!] = -\delta_{ij},$$

$$\left\langle \iota_{\frac{\partial}{\partial y^{i+n}}}, \mathcal{L}_{\frac{\partial}{\partial y^j}}; \omega_K \right\rangle = -[\![y^i, \mathrm{d}y^{j+n}]\!] = -\delta_{ij}.$$

All other terms vanish. Therefore, we obtain $\omega_K = \sum_{i=1}^{n} \mathrm{d}y^{-i} \wedge \mathrm{d}y^{i+n} + \mathrm{d}y^{i} \wedge \mathrm{d}y^{-i-n}$. $\square$

Remark 14.3. The Koszul–Schouten Poisson bracket is a particular case of odd Poisson bracket, also called antibracket or Buttin bracket. It can be shown that if the exterior differential, d, is a derivation of an odd Poisson bracket on (M, Ω_M), then it is a Koszul–Schouten bracket (see Beltrán and Monterde, 1995, Corollary 4.4). Moreover, the Koszul–Schouten bracket is an example of a Batalin–Vilkovisky structure on (M, Ω_M) (see Kosmann–Schwarzback and Monterde, 2000, Theorem 2.19).

14.2. Identification

Let L be a classical regular Lagrangian function on a manifold M. Then, it defines a nondegenerated Poisson bivector, P, on TM. Let us denote by ω_K the Koszul–Schouten symplectic form on the graded manifold $(TM, \Omega(TM))$ (i.e., $N = TM$). Therefore (TM, Ω_{TM}) together with the odd symplectic form ω_K, is an odd symplectic supermanifold.

On the other hand, let us consider the graded manifold (M, C^∞_M) and the same Lagrangian lifted to the graded first-order jet bundle. By applying the previous deductions to the corresponding Berezinian density and according to Theorem 13.1, we can build up a graded manifold, the manifold of solutions $(S, \mathcal{A}_S)$, together with a symplectic form ω_s.

In this section our main objective is to prove that (TM, Ω_{TM}) together with the odd symplectic form ω_K and $(S, \mathcal{A}_S)$ together with the symplectic form ω_s, are the same odd symplectic supermanifold.

Lemma 14.4. *Given a classical regular Lagrangian L, there is an isomorphism between the graded manifolds* $J^{1,1}(\mathbb{R}^{1|1}, (M, C^\infty_M))$ *and* $\mathbb{R}^{1|1} \times (TM, \Omega_{TM})$.

Proof: For both graded manifolds the base manifold is $\mathbb{R} \times TM = J^1(\mathbb{R}, M)$. Let us take on TM the system of classical canonical coordinates associated to the classical regular Lagrangian L, (x^i, p_i), $i = 1, \ldots, n$. According to this choice $(t, s, x^i, p_i, x^{-i}, p_{-i})$, $i = 1, \ldots, n$, is a system of graded coordinates on $\mathbb{R}^{1|1} \times (TM, \Omega_{TM})$ and $(t, s, x^i, p_i, x^i_s, p_{i,s})$, $i = 1, \ldots, n$, is a system of coordinates on $J^{1,1}(\mathbb{R}^{1|1}, (M, C^\infty_M))$.

The idea is to realize that jet coordinates of the kind x^i_s, $p_{i,s}$ behave like the differential 1-forms $x^{-i} = dx^i$, $p_{-i} = dp_i$.

We define the isomorphism between the two graded manifolds

$$\Phi\colon \mathbb{R}^{1|1} \times (TM, \Omega_{TM}) \to J^{1,1}\big(\mathbb{R}^{1|1}, \big(M, C^\infty_M\big)\big)$$

by setting

$$\Phi^*(t) = t, \qquad \Phi^*(s) = s,$$
$$\Phi^*(x^i) = x^i, \qquad \Phi^*(p_i) = p_i,$$
$$\Phi^*(x^{-i}) = x^i_s, \qquad \Phi^*(p_{-i}) = p_{i,s}.$$

It can be easily proved that this local definition is global. Indeed, if $x^i = x^i(z^1, \ldots, z^n)$ is a change of coordinates in M, then the corresponding changes of graded coordinates are given by

$$x^i_s = \frac{\partial x^i}{\partial z^j} z^j_s, \qquad p_i = \frac{\partial z^j}{\partial x^i} q_j,$$

where $q_j = \partial L/\partial z^j$. Moreover, recalling that x^{-i} denotes the odd generator $\mathrm{d}x^i$, we have

$$x^{-i} = \mathrm{d}x^i = \frac{\partial x^i}{\partial z^j}\, \mathrm{d}z^j = \frac{\partial x^i}{\partial z^j} z^{-j}.$$

This means

$$\Phi^*(x^{-i}) = \Phi^*\left(\frac{\partial x^i}{\partial z^j} z^{-j}\right) = \frac{\partial x^i}{\partial z^j}\Phi^*(z^{-j}) = \frac{\partial x^i}{\partial z^j} z^j_s = x^i_s.$$

And similarly for the coordinates $p_{i,s}$ and p_{-i}. Therefore, the map Φ is globally defined. $\square$

Theorem 14.5. *The graded symplectic manifolds (TM, Ω_{TM}) together with the Koszul–Schouten symplectic form ω_K and the manifold of solutions $(S, \mathcal{A}_S)$ together with the variational symplectic form ω_S, are isomorphic.*

Proof: In the graded coordinates introduced in Lemma 14.4, the Koszul odd symplectic form can be written as $\omega_K = \mathrm{d}x^{-i} \wedge \mathrm{d}p_i + \mathrm{d}x^i \wedge \mathrm{d}p_{-i}$, and the symplectic form on the graded manifold of solutions, $\omega_S = -(\mathrm{d}x^i_s \wedge \mathrm{d}p_i + \mathrm{d}x^i \wedge \mathrm{d}p_{i,s})$. Remark that $|x^i| = 0$ for the initial graded manifold is of graded dimension $(n, 0)$. Therefore, they are related by the isomorphism Φ: $\Phi^*(\omega_s) = -\omega_K$. $\square$

We have thus obtained the Koszul odd symplectic form, or equivalently the odd symplectic form defined by a nondegenerated Batalin–Vilkovisky structure, as a byproduct of a well known deduction of the symplectic structure associated to a regular Lagrangian, but adapting this deduction to the graded case.

14.3. The Hamiltonian Functions

Let L be a regular Lagrangian on M and let H be the corresponding classical Hamiltonian function; i.e., $H = L - x^i_t p_i$. Let ω be the associated symplectic form on TM and let X_H be the Hamiltonian vector field defined by $\iota_{X_H}\omega = dH$.

As above, let us study the same Lagrangian but from a graded point of view. The image by the isomorphism Φ of the t and s-Hamiltonian functions (10) associated to the graded problem are the following: $\Phi^*(H_t) = dH \in \Omega^1(TM)$, $\Phi^*(H_s) = -\omega \in \Omega^2(TM)$. Indeed,

$$\Phi^*(H_t) = \Phi^*\left(-x^i_{st}p_i - x^i_t p_{i,s} + \frac{dL}{ds}\right) = dx^i_t p_i - x^i_t dp_i + dL = dH,$$

and $\Phi^*(H_s) = \Phi^*(-x^i_s p_{i,s}) = -dx^i \wedge dp_i = -\omega$. Therefore, the graded Hamiltonian vector field with respect to the odd symplectic form ω_K (the Koszul–Schouten graded form) of these two graded functions are the following (see Proposition 2.6 and its remark in Beltrán and Monterde, 1995):

$$X^G_{\Phi^*(H_t)} = -\mathcal{L}_{X_H} \in \mathrm{Der}^0(\Omega(TM)), \qquad X^G_{\Phi^*(H_s)} = -d \in \mathrm{Der}^1(\Omega(TM)).$$

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